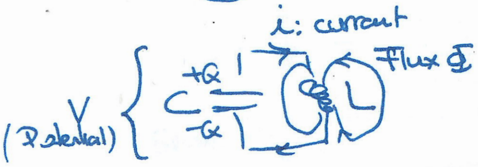
Quantum Optics

QUANTIZATION METHOD (REVISED)
(use of canonically conjugate variables)

check class. mech.

Quantizing Electrical Circuits (linear)



$\Phi_i = \int_{-\infty}^{t} V_i(t')\,dt'$ Flux

$q_i = \int_{-\infty}^{t} i_i(t')\,dt'$ CHARGE

Energies:

$U_{cap} = \frac{1}{2} C V^2$

$U_{ind} = \frac{1}{2} L I^2$

Lagrangian $\mathcal{L} = E_{kin} - E_{pot}$ (depends on choice of kinetic energy).

$$\mathcal{L} = \frac{1}{2} L I^2 - \frac{1}{2}\frac{q^2}{C}$$

$$\mathcal{L} = \underbrace{\frac{1}{2} L (\dot{q})^2}_{\text{Equivalent to kinetic energy}} - \underbrace{\frac{1}{2}\frac{q^2}{C}}_{\text{Potential Energy}}$$

Most generally one must use methods of NODES (Kirchhoff) and LOOPS ie $(\oint \vec{E}\,d\vec{l} = 0)$ and $\sum i_n = 0$ (current conservation)

Euler Lagrange equation:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_n} - \frac{\partial \mathcal{L}}{\partial \Phi_n} = 0}$$

here $\Phi_n = q$ and $\dot{\Phi}_n = \dot{q}$

i.e. $\frac{d}{dt}\left(\frac{2}{2} L \dot{q}\right) - \frac{q}{C} = 0$

$\Rightarrow L\ddot{q} - q/C = 0 \Rightarrow \ddot{q} = q \underbrace{\left(\frac{L}{C}\right)}_{\omega^2} \Rightarrow \omega_{LC} = \sqrt{\frac{1}{LC}}$ Eigenfrequency.

CONJUGATE VARIABLE TO CHARGE (Φ_n):

$$\boxed{q_n = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_n}}$$

Thus $q_n \hat{=} \frac{\partial \mathcal{L}}{\partial \dot{q}} = L \cdot \underset{i:\,\text{current}}{\dot{q}} = \Phi$ (Flux dans bobine)

→ This will be different depending on the choice of kin energy

$q_n \rightarrow \Phi$ conjugate momentum to charge is FLUX.

needed for QUANTIZATION RULE: $\boxed{[\hat{q}_n, \hat{\Phi}_n] = i\hbar}$ least action

HAMILTONIAN

$$H = (\Phi\,\dot{q}) - \mathcal{L} = \underbrace{\Phi}_{L\dot{q}}\dot{q} - \frac{1}{2}L(\dot{q})^2 + \frac{1}{2}\frac{q^2}{C}$$

$$H = \frac{\Phi^2}{2L} + \frac{1}{2C} q^2$$ expressed as canonically conjugate

Hamiltons equations:

$$\boxed{\begin{aligned}\dot{\Phi}_n &= \frac{\partial H}{\partial q_n}\\ \dot{q}_n &= -\frac{\partial H}{\partial \Phi_n}\end{aligned}}$$

This leads to: $q_u \hat{=} \Phi \;\wedge\; \Phi_u = \mathcal{H}$

$$\dot{\Phi}_u = q \ldots + \frac{\partial H}{\partial q} = \frac{\Phi}{L} = I = \dot{Q}$$
$$\dot{q}_u \hat{=} \dot{\Phi} = -\frac{\partial H}{\partial q} = -q \cdot C^{-1} = V$$

satisfies Hamilton's eqs.

~~Since~~ $V =$
QUANTIZATION proceeds by imposing on canonically conjugate variables:

$$[\hat{x}, \hat{p}] = -i\hbar \quad \text{(mech oscillator)}$$
$$\boxed{[\hat{\Phi}, \hat{q}] = -i\hbar}$$

One can rewrite the Hamiltonian as:

$$(*) \quad \hat{H} = \frac{\hbar\omega}{2}(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+) \quad \text{which results in:}$$

where now:

$$\hat{\Phi} = \sqrt{\frac{\hbar Z_0}{2}}\,(\hat{a} + \hat{a}^+)$$
$$\hat{Q} = \underbrace{\sqrt{\frac{\hbar}{2Z_0}}}_{\text{zero point charge fluctuation.}}\,\frac{(\hat{a} - \hat{a}^+)}{i}$$

$Z_0 = \sqrt{\frac{L}{C}}$ impedance.

Thus inserting expression yields $(*)$

or:
$$\boxed{\hat{a} = +i\frac{1}{\sqrt{2L\hbar\Omega}}\,\hat{\Phi} - \frac{1}{\sqrt{2C\hbar\Omega}}\,\hat{q}} \;\wedge\; [\hat{a}, \hat{a}^+] = 1$$

For Josephson junctions it is more convenient to use the flux Φ as momentum. (which before was $\dot{Q}$). Thus $V(t) = \dot{\Phi}$.

Hence: Capacitor (E_{kin})

$$E_{cap} = \frac{1}{2}C(\dot{\Phi})^2$$
$$E_{ind} = \frac{1}{2}\frac{\Phi^2}{L} \rightarrow \text{~~kinetic~~ energy}$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{1}{2}C(\dot{\Phi})^2 - \frac{1}{2L}\Phi^2}$$

Conjugate variable: $q_u = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_u} \Rightarrow q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$

$$\Rightarrow [\hat{q}, \hat{\Phi}] = +i\hbar$$
$$\text{(before } [\hat{\Phi}, \hat{Q}] = +i\hbar)$$

QUANTIZATION RULE AS APPLIED TO CANONICALLY CONJ. VARIABLES.
canonically conjugate variables.

Compute next the charge fluctuations in ground state $|0\rangle$

$$\langle 0|\Delta\hat{q}^2|0\rangle = \langle 0|\hat{q}^2|0\rangle \quad \text{since} \quad \langle 0|\hat{q}|0\rangle = 0$$

QUANTUM LC-OSCILLATOR

Expressing the operators:

$$\boxed{\begin{aligned} \hat{q} &= -i\, q_{ZPF}\,(\hat{a} - \hat{a}^{+}) \\ \hat{\Phi} &= \Phi_{ZPF}\,(\hat{a} + \hat{a}^{+}) \end{aligned}}$$

where $q_{ZPF} = \sqrt{\dfrac{C\hbar\omega}{2}} = \sqrt{\dfrac{\hbar}{2Z}}$ $\qquad Z = \sqrt{\left(\dfrac{C}{L}\right)} = \sqrt{\dfrac{L}{C}}$

impedance.

$\Phi_{ZPF} = \sqrt{\dfrac{L\hbar\omega}{2}} = \sqrt{\dfrac{\hbar Z}{2}}$

Thus:

$$\boxed{\begin{aligned} \langle 0|\, \hat{Q}^2 |0\rangle &= Q^2_{ZPF} \\ \langle 0|\, \hat{\Phi}^2 |0\rangle &= \Phi^2_{ZPF} \end{aligned}}$$

zero point fluctuations at the capacitor.

Thus: $\langle 0|\Delta \hat{u}^2|0\rangle = \dfrac{Q^2_{ZPF}}{e^2} = \dfrac{\hbar}{2Ze^2}$ charge fluctuations.